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On the Form of Solutions to the Linear Continuous Time Programming Problem and a Conjecture by Tyndall*

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This paper gives the general form of solutions to the linear continuous time programming problem and shows that the solutions are piecewise smooth.

1. INTRODUCTION

The continuous time linear programming problem is stated as follows: maximize

$$\int_0^T a(t)\, z(t)\, dt,$$

subject to

$$B(t)\, z(t) \leqslant c(t) + \int_0^t K(t, s)\, z(s)\, ds, \qquad t \in [0, t] \tag{1.1}$$

and

$$z(t) \geqslant 0, \qquad t \in [0, T].$$

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$B(t)$ and $K(t, s)$ are given $m \times n$ real matrices, $c(t)$ is an m-dimensional real vector, $a(t)$ is an n-dimensional real vector, and $z(t)$ is an unknown n-dimensional real vector.

By adding slack variables the first constraint can be made an equality constraint. The problem then becomes to maximize

$$\int_0^T a(t)\, z(t)\, dt$$

subject to

$$B^*(t)\, z^*(t) = c(t) + \int_0^t K(t, s)\, z(s)\, ds, \qquad t \in [0, T] \tag{1.2}$$

and

$$z^*(t) \geqslant 0, \qquad t \in [0, T].$$

Here

$$B^*(t) = [B(t), I]$$

and

$$z^*(t) = \begin{pmatrix} z(t) \\ z^s(t) \end{pmatrix},$$

where I is the $m \times m$ identity and $z^s(t)$ is an unknown m-dimensional real vector of slack variables.

Recently, Jóhannesson and Hanson [2] showed that the optimum solution to such a system occurs when at most m of the $m+n$ elements of $z^*(t)$ are positive. The main theorem of that paper involved a condition on the dual problem associated with (1.1). The dual is to minimize

$$\int_0^T w(t)\, c(t)\, dt$$

subject to

$$w(t)\, B(t) \geqslant a(t) + \int_t^T w(s)\, K(s, t)\, ds, \qquad t \in [0, T] \tag{1.3}$$

and

$$w(t) \geqslant 0, \qquad t \in [0, T].$$

This dual problem can be written by introducing slack variables. The constraints will then become

$$w^*(t)\, B^{\oplus}(t) = a(t) + \int_t^T w(s)\, K(s, t)\, ds, \qquad t \in [0, T],$$

$$w^*(t) \geqslant 0, \qquad t \in [0, T].$$

Here $w(t)$ is an m-dimensional real vector and

$$w^*(t) = (w(t), w^s(t)),$$

$$B^{\oplus}(t) = \begin{bmatrix} B(t) \\ -I \end{bmatrix},$$

where $w^s(t)$ is an n-dimensional vector of slacks and I is an $n \times n$ identity matrix.

Define

$$L_j = w^*(t)\, B^{\oplus}(t)_{*j} - a_j(t) - \int_t^T w(s)\, K(s, t)_{*j}\, ds$$

with $B^{\oplus}(t)$ as above, $B^{\oplus}_{*j}(t)$ is the jth column of $B^{\oplus}(t)$, $K(s, t)_{*j}$ the jth column of $K(s, t)$, and $a_j(t)$ the jth component of $a(t)$. Then L_j represents the jth constraint in the dual. Jóhannesson and Hanson obtained the following

THEOREM 1. *If at optimum, $\hat{w}(t)$, the L_j are linearly independent almost everywhere, any optimum solution to* (1.1) *will contain at most m positive variables for almost all $t \in [0, T]$.*

This theorem will be used to obtain a general analytic solution to the problem when B and K are time independent. That in turn will allow the confirmation of a conjecture made by Tyndall [3] on the nature of the solutions.

The construction of a solution requires some results from the theory of systems of differential equations. These are supplied in the next section.

2. Results from the Theory on Differential Equations

The discussion here follows that of Bellman [1].

Let $z(t)$ be a vector function. Consider the problem of solving an equation of the form

$$\frac{dz(t)}{dt} = Az(t) \tag{2.1}$$

subject to

$$z(0) = c.$$

A is a square matrix of constants and c is a vector. The notation $dz(t)/dt$ is used for term by term differentiation, that is,

$$\left(\frac{dz(t)}{dt}\right)' = \left(\frac{dz_1(t)}{dt}, \frac{dz_2(t)}{dt}, ..., \frac{dz_m(t)}{dt}\right).$$

Anagolously to ordinary differential equations, the solution will be of exponential form. However, the following definition of the *matrix exponential* is needed:

$$\exp(At) = e^{At} \equiv I + At + \cdots + A^n \frac{t^n}{n!} + \cdots.$$

Bellman [1] provides the following theorem.

THEOREM 2. *The matrix series e^{At} exists for all A for any fixed value of t and for all t for any fixed A. It converges uniformly in any finite region of the complex t plane.*

It is easy to see that $z(t) = e^{At}c$ is a solution to (2.1) Bellman [1] proves that the solution is unique.

By using the matrix exponential one can now find solutions to the original problem.

3. Form of the Optimal Solution

From the results of Jóhannesson and Hanson [2] it is clear that we can concentrate on basic feasible solutions, that is, solutions with at most m positive variables. In practice it is common to see solutions which are not continuous for all $t \in [0, T]$. One such case can be found in [2]. Therefore the matrix exponential results have to be used piecewise on subintervals. It is important to stress that the main difference between the programming problem and the differential equation problem is that the former requires nonnegativity but not necessarily continuity. In what follows the model with the slack variables (1.2) is assumed, so that the first constraint is an equality constraint. The matrices B and K are independent of time.

One proceeds as follows. First the matrix $B^* = [B \mid I]$ is partitioned in such a way as to select the columns of B^* corresponding to the optimal solution, that is, $B^* = [B_1 \mid B_1^*]$. Here, B_1 corresponds to the m basic variables and B_1^* corresponds to the variables which are zero over the first time interval, say, $[0, t_1]$. It should be noted that in general the basic variables are not necessarily the first m variables. When B_1 is known, the solution will agree with the solution to the system

$$B_1 \bar{z}_1(t) = c(t) + \int_0^t K_1 \bar{z}_1(s)\, ds, \tag{3.1}$$

on the interval $[0, t_1]$. Note that B_1 is an $m \times m$ matrix. In (3.1) $\bar{z}_1(t)$ is the vector of the m basic variables. K_1 is an $m \times m$ matrix, which contains the columns of K corresponding to the basic original variables, and columns of zeros for basic slack variables. Say, for example, that the original system has two inequalities and two variables. Then $z_1(1)$ and $z_2(t)$ are original variables, $z_3(t)$ and $z_4(t)$ are slacks. If $K = [\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}]$, and $z_1(t)$ and $z_3(t)$ are basic on $[0, t_1]$, then $\bar{z}_1(t) = [z_1(t), z_3(t)]$ and $K_1 = [\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}]$. By assuming that B_1 is invertible and using successive approximation we get the following solution to (3.1):

$$\bar{z}_1(t) = B_1^{-1} c(t) + \int_0^t B_1^{-1} K_1 B_1^{-1} \exp(K_1 B_1^{-1}(t-u))\, c(u)\, du,$$

for $t \in [0, t_1]$. This can be verified by substitution. Here $\exp(\cdot)$ is the matrix exponential from Section 2.

To get a solution for the original system (1.2), n rows of zeros corresponding to the variables not in the basis are inserted into B_1^{-1}, thus forming a new matrix G_1.

Say, in the previous example, where $z_1(t)$ and $z_3(t)$ were basic on $[0, t_1]$, that B_1^{-1} was found to be $[\begin{smallmatrix} 1 & 5 \\ 6 & 1 \end{smallmatrix}]$. Then

$$G_1 = \begin{bmatrix} 1 & 5 \\ 0 & 0 \\ 6 & 1 \\ 0 & 0 \end{bmatrix}.$$

Also define $\bar{K} = [K \mid 0]$ where 0 is an $m \times m$ matrix of zeros. In the previous example $K = [\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}]$ so $\bar{K} = [\begin{smallmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{smallmatrix}]$.

Then a solution to (1.2) is given by

$$\hat{z}_1(t) = G_1 c(t) + \int_0^t G_1 \bar{K} G_1 \exp(\bar{K} G_1 (t-u))\, c(u)\, du$$

for $t \in [0, t_1]$. This solution agrees with the one previously obtained but has set nonbasic variables equal to zero.

Now the form of the solution is known for the interval $[0, t_1]$. The point

t_1 is called the first join point. This can now be used to find the form of the solution on the interval $[t_1, t_2]$. Define $h(t_1)$ as follows:

$$h(t_1) = \int_0^{t_1} \bar{K}\hat{z}_1(s)\,ds$$
$$= \int_0^{t_1} \bar{K}(G_1 c(t) + \int_0^t G_1 \bar{K} G_1 \exp(\bar{K}G_1(t-u))\,c(u)\,du)\,dt.$$

As before, one can partition the matrix $B^* = [B_2 \mid B_2^*]$, where $B_2 \neq B_1$ corresponds to the basic solution on $(t_1, t_2]$. Then on the interval $(t_1, t_2]$ the solution agrees with the solution to the system

$$B_2 \bar{z}_2(t) = c(t) + h(t_1) + \int_{t_1}^t K_2 \bar{z}_2(s)\,ds,$$

where $\bar{z}_2(t)$ and K_2 correspond to the new basic solution. By following the same path as before one finds the solution

$$\hat{z}_2(t) = G_2(c(t) + h(t_1)) + \int_{t_1}^t G_2 \bar{K} G_2 \exp(\bar{K}G_2(t-u))(c(u) + h(t_1))\,du$$

on $(t_1, t_2]$. Hence, one sees that the solution on this time interval depends on the previous solution through the term $h(t_1)$. By continuing in this manner we establish

THEOREM 3. *The form of the solution between the join t_{j-1} and t_j is*

$$\hat{z}_j(t) = G_j(c(t) + h(t_{j-1})) + \int_{t_{j-1}}^t G_j \bar{K} G_j \exp(\bar{K}G_j(t-u))(c(u) + h(t_{j-1}))\,du \tag{3.2}$$

where

$$h(t_i) = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \bar{K}\hat{z}_j(s)\,ds \tag{3.3}$$

and

$$t_0 = 0, \qquad h(t_0) = 0,$$

provided that the B_j corresponding to the optimal solution are all invertible.

4. Tyndall's Conjecture

In one of the early papers on continuous time programming, Tyndall [3] conjectured that when B and K are independent of time and $c(t)$ is smooth, then the solutions to (1.1) would be piecewise smooth functions. This conjecture has now been confirmed. In Section 3 the form of the solution is shown to be given by (3.2). Since $c(t)$ is a smooth function, this form can be differentiated between join points. We state this formally as

THEOREM 4. *When K and B are independent of time and $c(t)$ is smooth, then the solution to* (1.1) *is piecewise smooth. It is given by* (3.2) *and the derivative between the join points t_{j-1} and t_j is*

$$\frac{d\hat{z}(t)}{dt} = G_j \frac{dc(t)}{dt} + G_j\bar{K}G_j(c(t) + h(t_{j-1}))$$
$$+ \int_{t_{j-1}}^{t} G_j\bar{K}G_j\bar{K}G_j \exp(\bar{K}G_j(t-u))(c(u) + h(t_{j-1}))\, du$$

where $h(t_j)$ is given by (3.3).

References

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